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Induced W_∞ gravity as a WZNW model

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We derive the explicit form of the Wess–Zumino quantum effective action of chiral W_∞-symmetric system of matter fields coupled to a general chiral W_∞-gravity background. It is expressed as a geometric action on a coadjoint orbit of the deformed group of area-preserving diffeomorphisms on cylinder whose underlying Lie algebra is the centrally-extended algebra of symbols of differential operators on the circle. Also, we present a systematic derivation, in terms of symbols, of the "hidden" $SL(\infty;\mathbb{R})$ Kac–Moody currents and the associated $SL(\infty,\mathbb{R})$ Sugawara form of energy–momentum tensor component T_{++} as a consequence of the $SL(\infty;\mathbb{R})$ stationary subgroup of the relevant W_∞ coadjoint orbit.

1. Introduction

The infinite-dimensional Lie algebra W_∞ (and its generalizations $W_{1+\infty}$ etc.) [1–3] are nontrivial "large N" limits of the associative, but *non*-Lie, conformal W_N algebras [4]. They arise in various problems of two-dimensional physics. The list of their principal applications includes self-dual gravity [5], first hamiltonian structure of integrable KP hierarchy [6], string field actions in the collective field theory approach [7], conformal affine Toda theories [8]. One of the most remarkable manifestations of W_∞-type algebras is the recent discovery of a subalgebra of their "classical" limit w_∞ (the algebra of area-preserving diffeomorphisms) in $c=1$ string theory as symmetry algebra of the special discrete states [9] or as the algebra of infinitesimal deformations of the ground ring [10]. Also, it is worth noting that similar algebras are found also in $D=2$ quasitopological models, such as $D=2$ Yang–Mills [11], where the metric dependence of the partition function degenerates into a dependence on the area only.

It is well known in the mathematical literature [12], that the family of possible deformations $W_\infty(q)$ of the initial "classical" w_∞ depends on a single parameter q and that, for each fixed value of q, $W_\infty(q)$ possesses an one-dimensional cohomology with values of $\mathbb{R}$. In particular, for $q=1$ one finds that $W_\infty(1)\simeq\widetilde{\mathcal{DOP}}(S^1)$ – the centrally extended algebra of differential operators on the circle, which was recently studied in ref. [13]. The equivalence of $\widetilde{\mathcal{DOP}}(S^1)$ to the original definition of $W_\infty(1)$ [1,3] was explicitly demonstrated in ref. [14].

In this letter we first derive a WZNW field-theory action $W_{DOP(S^1)}[g]$ on a generic coadjoint orbit of the group $G=DOP(S^1)$. The elements $g(\xi,x;t)$ of this group for fixed time t are symbols of exponentiated differential operators on S^1 and in this sense $DOP(S^1)$ is the formal Lie group corresponding to the Lie algebra $\widetilde{\mathcal{DOP}}(S^1)$. As it was shown in ref. [15], the Legendre transform $\Gamma[g]=-W[g^{-1}]$ of a group coadjoint orbit action $W[g]$ for a general infinite-dimensional group G provides the exact solution for the quantum effective

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action of matter fields possessing an infinite-dimensional Noether symmetry group G_0 – the "classical" undeformed version of the group G. Thus, our WZNW action $W_{DOP(S^1)}[g]$ is the explicit field-theoretic expression of the induced W_∞-gravity effective action. In particular, we show that $W_{DOP(S^1)}[g]$ reduces to the well-known Polyakov's WZNW action of induced $D=2$ gravity in the light-cone gauge [16] when restricting the WZNW field $g(\xi, x; t)$ to the Virasoro subgroup of $DOP(S^1)$. Furthermore, the appearance of the "hidden" $SL(\infty; \mathbb{R})$ Kac-Moody symmetry and the associated $SL(\infty; \mathbb{R})$ Sugawara form of the T_{++} component of the energy-momentum tensor are shown to be natural consequences of the $SL(\infty; \mathbb{R})$ stationary subgroup of the pertinent $DOP(S^1)$ coadjoint orbit. Also, we present WZNW field-theoretic expressions in terms of $g(\xi, x; t)$ for the "hidden" currents and T_{++}.

2. Basic ingredients

The object of primary interest is the infinite-dimensional Lie algebra $\mathcal{G} = \mathcal{DOP}(S^1)$ of symbols of differential operators [#1] on the circle S^1 with vanishing zero-order part $\tilde{\mathcal{G}} = \{X \equiv X(\xi, x) = \sum_{k\geqslant 1} \xi^k X_k(x)\}$. For any pair $X, Y \in \mathcal{G} = \mathcal{DOP}(S^1)$ the Lie commutator is given in terms of the associative (and *non*-commutative) symbol product denoted henceforth by a circle $\circ$:

$$[X, Y] \equiv X\circ Y - Y\circ X , \quad X\circ Y \equiv X(\xi, x) \exp(\overleftarrow{\partial}_\xi \overrightarrow{\partial}_x) Y(\xi, x) . \tag{1}$$

In order to determine the dual space $\mathcal{G}^* = \mathcal{DOP}^*(S^1)$, let us consider the space $\Psi\mathcal{DO}(S^1) = \{U \equiv U(\xi, x) = \sum_{k=1}^{\infty} \xi^{-k} U_k(x)\}$ of all purely pseudodifferential symbols [17] on S^1 and the following bilinear form on $\Psi\mathcal{DO}(S^1) \otimes \mathcal{DOP}(S^1)$:

$$\langle U | X \rangle \equiv \int dx \, \mathrm{Res}_\xi \, U\circ X = \int dx \, \mathrm{Res}_\xi \, [\exp(-\partial_x \partial_\xi) U(\xi, x)] \, X(\xi, x) . \tag{2}$$

The last equality in (2) is due to the vanishing of total derivatives with respect to the measure $\int dx \, \mathrm{Res}_\xi$, and $\mathrm{Res}_\xi \, U(\xi, x) = U_1(x)$. From (2) we conclude that any pseudodifferential symbol of the form $U^{(0)} = \exp(\partial_x \partial_\xi)[(1/\xi) u(x)]$ is "orthogonal" to any differential symbol $X \in \mathcal{DOP}(S^1)$, i.e. $\langle U^{(0)} | X \rangle = 0$. Thus, the dual space $\mathcal{G}^* = \mathcal{DOP}^*(S^1)$ can be defined as the factor space $\Psi\mathcal{DO}(S^1) \setminus [\exp(\partial_x \partial_\xi)(1/\xi) u(x)]$ with respect to the "zero" pseudodifferential symbols. In particular, we shall adopt the definition

$$\mathcal{G}^* = \left[U_*; \, U_*(\xi, x) = U(\xi, x) - \exp(\partial_x \partial_\xi) \left(\frac{1}{\xi} \mathrm{Res}_\xi \, U(\xi, x) \right) \text{for } \forall \, U \in \Psi\mathcal{DO} \right] . \tag{3}$$

Having the bilinear form (2) one can define the coadjoint action of $\mathcal{G}$ on $\mathcal{G}^*$ via

$$\langle \mathrm{ad}^*(X) U | Y \rangle = -\langle U | [X, Y] \rangle , \quad (\mathrm{ad}^*(X) U)(\xi, x) \equiv [X, U]_* . \tag{4}$$

Here and in what follows, the subscript $(-)$ indicates taking the part of the symbol containing all negative powers in the ξ-expansion, whereas the subscript $*$ indicates projecting of the symbol on the dual space (3). The Jacobi identity for the coadjoint action $\mathrm{ad}^*(\ )$ (4) is fulfilled due to the following important property:

$$\left[X, \exp(\partial_x \partial_\xi) \left(\frac{1}{\xi} u(x) \right) \right]_- = \exp(\partial_x \partial_\xi) \left(\frac{1}{\xi} \mathrm{Res}_\xi \left[X, \exp(\partial_x \partial_\xi) \frac{u(x)}{\xi} \right] \right) , \tag{5}$$

i.e., the coadjoint action of $\mathcal{DOP}(S^1)$ on $\Psi\mathcal{DO}(S^1)$ maps "zero" pseudodifferential symbols into "zero" ones.

The central extension in $\tilde{\mathcal{G}} \equiv \widetilde{\mathcal{DOP}}(S^1) = \mathcal{DOP}(S^1) \oplus \mathbb{R}$ is given by the two-cocycle $\omega(X, Y) = -(1/4\pi) \langle \hat{s}(X) | Y \rangle$, where the cocycle operator $\hat{s}: \mathcal{G} \to \mathcal{G}^*$ explicitly reads [13]

[#1] Let us recall [17] the correspondence between (pseudo)differential operators and symbols: $X(\xi, x) = \sum_k \xi^k X_k(x) \leftrightarrow \hat{X} = \sum_k X_k(x)(-i\partial_x)^k$.

$$\hat{s}(X)=[X,\ln\xi]_* . \tag{6}$$

Let us now consider the Lie group $G=\mathrm{DOP}(S^1)$ defined as exponentiation of the Lie algebra $\mathscr{G}$ of symbols of differential operators on S^1:

$$G=\left(g(\xi,x)=\mathrm{Exp}\,[X(\xi,x)]\equiv\sum_{N=0}^{\infty}\frac{1}{N!}X(\xi,x)\circ X(\xi,x)\circ\ldots\circ X(\xi,x)\right), \tag{7}$$

and the group multiplication is just the symbol product $g\circ h$. The adjoint and coadjoint action of $G=\mathrm{DOP}(S^1)$ on the Lie algebra $\mathscr{DOP}(S^1)$ and its dual space $\mathscr{DOP}^*(S^1)$, respectively, is given as

$$[\mathrm{Ad}(g)X]=g\circ X\circ g^{-1}, \quad (\mathrm{Ad}^*(g)U)=(g\circ X\circ g^{-1})_* . \tag{8}$$

The group property of (8) $\mathrm{Ad}^*(g\circ h)=\mathrm{Ad}^*(g)\mathrm{Ad}^*(h)$ easily follows from the "exponentiated" form of the identity (5).

After these preliminaries we are ready to introduce the two interrelated fundamental objects $S[g]$ and $Y[g]$ entering the construction of the geometric action on a coadjoint orbit of G. To this end we shall follow the general formalism for geometric actions on coadjoint orbits of arbitrary infinite-dimensional groups with central extensions proposed in refs. [18,19]. Namely, $S[g]$ is a nontrivial $\mathscr{G}^*$-valued one-cocycle on the group G (also called finite "anomaly" or generalized schwarzian), whose infinitesimal form is expressed through the Lie-algebra $\mathscr{G}$ cocycle operator $\hat{s}(\;)$ (6) (infinitesimal "anomaly"):

$$S[g\circ h]=S[g]+\mathrm{Ad}^*(g)S[h], \quad \frac{\mathrm{d}}{\mathrm{d}t}S[\mathrm{Exp}(tX)]\Big|_{t=0}=\hat{s}(X) . \tag{9}$$

The explicit solution of eq. (9) reads

$$S[g]=-([\ln\xi,g]\circ g^{-1})_* . \tag{10}$$

Further, $Y[g]$ is a $\mathscr{G}$-valued one-form on the group manifold which is related to the $\mathscr{G}^*$-valued group one-cocycle $S[g]$ via the following basic exterior-derivative equation:

$$\mathrm{d}S[g]=-\mathrm{Ad}^*(g)\hat{s}(Y[g^{-1}]) . \tag{11}$$

The integrability condition for (11) implies that the one-form $Y[g]$ satisfies the Maurer–Cartan equation and that it is a $\mathscr{DOP}(S^1)$-valued group one-cocycle:

$$\mathrm{d}Y[g]=\tfrac{1}{2}[Y[g],Y[g]], \quad Y[g\circ h]=Y[g]+\mathrm{Ad}(g)Y[h] . \tag{12}$$

From (6) and (10)–(12) one easily finds

$$Y[g]=\mathrm{d}g(\xi,x)\circ g^{-1}(\xi,x) . \tag{13}$$

At this point it would be instructive to explicate formulas (6), (10) and (13) when the elements of $G=\mathrm{DOP}(S^1)$ and $\mathscr{G}=\mathscr{DOP}(S^1)$ are restricted to the Virasoro subgroup (subalgebra, respectively):

$$X(\xi,x)=\xi\omega(x)\leftrightarrow\omega(x)\partial_x\in\mathrm{Vir},$$

$$g(\xi,x)=\mathrm{Exp}[\xi\omega(x)]\leftrightarrow F(x)\equiv\exp[\omega(x)\partial_x]x\in\mathrm{Diff}(S^1) . \tag{14}$$

Substituting (14) into (6), (10) and (13), one obtains

$$Y[g]\,|_{g(\xi,x)=\mathrm{Exp}[\xi\omega(x)]}=\xi\frac{\mathrm{d}F(x)}{\partial_x F(x)}, \quad \hat{s}(X)=[\xi\omega(x),\ln\xi]_*=-\tfrac{1}{6}\xi^{-2}\,\partial_x^3\omega(x)+\ldots,$$

$$S[g]\,|_{g(\xi,x)=\mathrm{Exp}[\xi\omega(x)]}=-\tfrac{1}{6}\xi^{-2}\left[\frac{\partial_x^3 F}{\partial_x F}-\frac{3}{2}\left(\frac{\partial_x^2 F}{\partial_x F}\right)^2\right]+\ldots . \tag{15}$$

The dots in (15) indicate higher order terms $O(\xi^k)$, $k\geqslant 3$, which do not contribute in bilinear forms with elements of Vir (14).

3. WZNW action of W_∞ gravity

According to the general theory of group coadjoint orbits [20], a generic coadjoint orbit $\mathcal{O}_{(U_0,c)}$ of G passing through a point (U_0, c) in the extended dual space $\tilde{\mathcal{G}}^* = \mathcal{G}^* \oplus \mathbb{R}$:

$$\mathcal{O}_{(U_0,c)} \equiv \{ (U(g), c) \in \tilde{\mathcal{G}}^* ; \quad U(g) = \mathrm{Ad}^*(g) U_0 + cS[g] \} \tag{16}$$

has a structure of a phase space of an (infinite-dimensional) hamiltonian system. Its dynamics is governed by the following lagrangian geometric action written solely in terms of the interrelated fundamental group and algebra cocycles $S[g]$, $Y[g]$, $\hat{s}(\ )$ (cf. eqs. (6), (9)–(13)) [18,19]:

$$W[g] = \int_{\mathcal{L}} \langle U_0 | Y[g^{-1}] \rangle - c \int [\langle S[g] | Y[g] \rangle - \tfrac{1}{2} \mathrm{d}^{-1} (\langle \hat{s}(Y[g]) | Y[g] \rangle)] \,. \tag{17}$$

The integral in (17) is over one-dimensional curve $\mathcal{L}$ on the phase space $\mathcal{O}_{(U_0,c)}$ (16) with a "time-evolution" parameter t. Along the curve $\mathcal{L}$ the exterior derivative becomes $\mathrm{d} = \mathrm{d}t\, \partial_t$. Also, d^{-1} denotes the cohomological operator of Novikov [21] – the inverse of the exterior derivative, defining the customary multi-valued term present in any geometric action on a group coadjoint orbit.

In the present case of $G = \mathrm{DOP}(S^1)$, the co-orbit action (17) takes the following explicit form, which (as discussed in section 1) is precisely the Wess–Zumino action for induced W_∞-gravity (the explicit dependence of symbols on $(\xi, x; t)$ will in general be suppressed below):

$$W[g] = -\int \mathrm{d}t\, \mathrm{d}x\, \mathrm{Res}_\xi\, U_0 \circ g^{-1} \circ \partial_t g$$
$$+ \frac{c}{4\pi} \int_{\mathcal{L}} \int \mathrm{d}x\, \mathrm{Res}_\xi ([\ln \xi, g] \circ g^{-1} \circ \partial_t g \circ g^{-1} - \tfrac{1}{2} \mathrm{d}^{-1} \{ [\ln \xi, \mathrm{d}g \circ g^{-1}] \wedge (\mathrm{d}g \circ g^{-1}) \}) \,. \tag{18}$$

The physical meaning of the first term on the RHS of (18) is that of coupling of the chiral W_∞ Wess–Zumino field $g = g(\xi, x; t)$ to a chiral W_∞-gravity "background". For simplicity, we shall consider henceforth the case $U_0 = 0$.

It is straightforward to obtain, upon substitution of eqs. (14), (15), that the restriction of $g(\xi, x; t)$ to the Virasoro subgroup reduces the W_∞ Wess–Zumino action (18) to the well-known Polyakov's Wess–Zumino action of induced $D=2$ gravity [16,22].

The group cocycle properties (eqs. (9), (12)) of $S[g]$ (10) and $Y[g]$ (13) imply the following fundamental group composition law for the W_∞ geometric action (18):

$$W[g \circ h] = W[g] + W[h] - \frac{c}{4\pi} \int \mathrm{d}t\, \mathrm{d}x\, \mathrm{Res}_\xi ([\ln \xi, h] \circ h^{-1} \circ g^{-1} \circ \partial_t g) \,. \tag{19}$$

Eq. (19) is a particular case for W_∞ of the group composition law for geometric actions on coadjoint orbits of arbitrary infinite-dimensional groups with central extensions [19]. It generalizes the famous Polyakov–Wiegmann group composition law [23] for ordinary $D=2$ WZNW models.

Using the general formalism for co-orbit actions in refs. [18,19] we find that the basic Poisson brackets for $S[g]$ (10) following from the action (18) read

$$\{ S[g](\xi, x), S[g](\eta, y) \}_{\mathrm{PB}} = [S[g](\xi, x) + \ln \xi, \delta_{\mathrm{DOP}}(y, \eta; x, \xi)]_* \,, \tag{20}$$

where $\delta_{\rm DOP}(\ ;\ ) \in \mathscr{G}^* \otimes \mathscr{G}$ denotes the kernel of the δ-function on the space of differential operator symbols:

$$\delta_{\rm DOP}(x, \xi; y, \eta) = \exp(\partial_x \partial_\xi) \left(\sum_{k=1}^{\infty} \xi^{-(k+1)} \eta^k \delta(x-y) \right). \tag{21}$$

Eq. (20) is a succinct expression of the Poisson-bracket realization of W_∞, which becomes manifest by rewriting (20) in the equivalent form:

$$\{\langle S[g] | X\rangle, \langle S[g] | Y\rangle\}_{\rm PB} = -\langle S[g] | [X, Y]\rangle + \langle \hat{s}(X) | Y\rangle \tag{22}$$

for arbitrary fixed $X, Y \in \mathscr{G} = \mathscr{DOP}(S^1)$. Alternatively, substituting into (20) (or (22)) the ξ-expansion of the pseudodifferential symbol $S[g](\xi, x) = \sum_{r\geq 2} \xi^{-r} S_r(x)$, one recovers the Poisson-bracket commutation relations for W_∞ among the component fields $S_r(x)$ in the basis of ref. [14] (which is a "rotation" of the more customary W_∞ basis of ref. [2]).

In particular, for the component field $S_2(x) \equiv (4\pi/c) T_{--}(x)$ (the energy–momentum tensor component, cf. (15)) one gets from (20) the Poisson-bracket realization of the Virasoro algebra:

$$\{S_2(x), S_2(y)\}_{\rm PB} = -\frac{4\pi}{c} [2S_2(x)\, \partial_x \delta(x-y) + \partial_x S_2(x) \delta(x-y) + \tfrac{1}{6}\, \partial_x^3 \delta(x-y)]\,. \tag{23}$$

The higher component fields $S_r(x)$, $r = 3, 4, \ldots$ turn out to be quasi-primary conformal fields of spin r. The genuine primary fields $\mathscr{W}_r(x)$ ($r \geq 3$) are obtained from $S_r(x)$ by adding derivatives of the lower spin fields $S_q(x)$ ($2 \leq q \leq r-1$). For instance, for $\mathscr{W}_3(x) = S_3(x) - \frac{3}{2}\, \partial_x S_2(x)$, eq. (20) yields

$$\{S_2(x), \mathscr{W}_3(y)\}_{\rm PB} = -\frac{4\pi}{c} [3\mathscr{W}_3(x)\, \partial_x \delta(x-y) + 2\, \partial_x \mathscr{W}_3(x)\, \delta(x-y)]\,. \tag{24}$$

4. Noether and "hidden" symmetries of W_∞ gravity

The general group composition law (19) contains the whole information about the symmetries of the W_∞ geometric action (18). First, let us consider arbitrary infinitesimal left group translation. The corresponding variation of the action (18) is straightforwardly obtained from (19)

$$\delta_\epsilon^{\rm L} W[g] \equiv W[(\mathbb{1} + \epsilon) \circ g] - W[g] = \frac{c}{4\pi} \int {\rm d}t\, {\rm d}x\, {\rm Res}_\xi \{([\ln \xi, g] \circ g^{-1})_* \circ \partial_t \epsilon\}\,. \tag{25}$$

From (25) one finds that (18) is invariant under t-independent left group translations and the associated Noether conserved current is the generalized "schwarzian" $S[g]$ (10) whose components are the (quasi)primary conformal fields $S_r(x; t)$ of spin r.

Next, let us consider arbitrary right group translation. Now, from (19) the variation of the W_∞ action (18) is given by

$$\delta_\zeta^{\rm R} W[g] \equiv W[g \circ (\mathbb{1} + \zeta)] - W[g] = -\frac{c}{4\pi} \int {\rm d}t\, {\rm d}x\, {\rm Res}_\xi ([\ln \xi, \zeta]_* \circ Y_t(g^{-1})) \tag{26}$$

$$= \frac{c}{4\pi} \int {\rm d}t\, {\rm d}x\, {\rm Res}_\xi ([\ln \xi, Y_t(g^{-1})]_* \circ \zeta)\,, \tag{27}$$

where $Y_t(g^{-1})$ denotes the Maurer–Cartan gauge field:

$$Y_t(g^{-1}) = -g^{-1} \circ \partial_t g\,. \tag{28}$$

Equality (27) implies the equations of motion [#2]:

$$\hat{s}(Y_t(g^{-1}))|_{\text{on-shell}} = 0 . \tag{29}$$

As a matter of fact, the *off*-shell relation (11) exhibits the full equivalence between the Noether conservation law $\partial_t S[g] = 0$ (25) and the equations of motion (29).

On the other hand, equality (26) shows that the W_∞ geometric action (18) is *gauge*-invariant under arbitrary *time-dependent* infinitesimal right group translations $g(\xi, x; t) \to g(\xi, x; t) \circ (1 + \tilde{\zeta}(\xi, x; t))$ which satisfy

$$\hat{s}(\tilde{\zeta}) \equiv -[\ln \xi, \tilde{\zeta}]_* = 0 . \tag{30}$$

For finite right group translations $k = \text{Exp}(\tilde{\zeta})$ the integrated form of (30) reads

$$S[k] \equiv -([\ln \xi, k] \circ k^{-1})_* = 0 . \tag{31}$$

The solutions of eqs. (30) and (31) form a subalgebra in $\mathscr{DOP}(S^1)$, and a subgroup in $\text{DOP}(S^1)$, respectively. From (16) one immediately concludes that the latter subgroup,

$$G_{\text{stat}} = \{k;\, S[k] = 0\} , \tag{32}$$

is precisely the stationary subgroup of the underlying coadjoint orbit $\mathcal{O}_{(U_0=0,c)}$. The Lie algebra of (32),

$$\mathscr{G}_{\text{stat}} = \{\tilde{\zeta};\, \hat{s}(\tilde{\zeta}) \equiv -[\ln \xi, \tilde{\zeta}]_* = 0\} , \tag{33}$$

is the maximal centerless ("anomaly-free") subalgebra of $\widetilde{\mathscr{DOP}}(S^1)$, on which the cocycle (6) vanishes: $\omega(\tilde{\zeta}_1, \tilde{\zeta}_2) = -\langle \hat{s}(\tilde{\zeta}_1) | \tilde{\zeta}_2 \rangle = 0$ for any pair $\tilde{\zeta}_{1,2} \in \mathscr{G}_{\text{stat}}$.

The full set of linearly independent solutions $\{\zeta^{(l,m)}(\xi, x)\}$ of $\hat{s}(\tilde{\zeta}) = 0$, comprising a basis in $\mathscr{G}_{\text{stat}}$ (33), can be written in the form

$$\zeta^{(l,m)}(\xi, x) = \sum_{q=1}^{l} \binom{l}{q} \frac{(l-1)!(l+q)!}{(q-1)!(2l)!} \frac{\xi^q x^{q+m}}{\Gamma(q+m+1)} , \tag{34}$$

where $l = 1, 2, \ldots$, and $m = -l, -l+1, \ldots, l-1, l$.

The basis (34) identifies the stationary subalgebra $\mathscr{G}_{\text{stat}}$ (33) as the infinite-dimensional algebra $\text{sl}(\infty; \mathbb{R})$. Namely, $\mathscr{G}_{\text{stat}}$ decomposes (as a vector space) into a direct sum of irreducible representations $\mathscr{V}^{(l)}_{\text{sl}(2)}$ of its $\text{sl}(2; \mathbb{R})$ subalgebra with spin l and unit multiplicity: $\mathscr{G}_{\text{stat}} = \oplus_{l=1}^{\infty} \mathscr{V}^{(l)}_{\text{sl}(2)}$. This $\text{sl}(2; \mathbb{R})$ subalgebra is generated by the symbols $2\zeta^{(1,1)} = \xi x^2$, $\zeta^{(1,0)} = \xi x$ and $\zeta^{(1,-1)} = \xi$. The subspaces $\mathscr{V}^{(l)}_{\text{sl}(2)}$ are spanned by the symbols $\{\zeta^{(l,m)};\, l = \text{fixed},\, |m| \leqslant l\}$ with $\zeta^{(l,l)}$ being the highest-weight vectors:

$$[\xi x^2, \zeta^{(l,l)}] = 0 , \quad [\xi x, \zeta^{(l,m)}] = m\zeta^{(l,m)} , \quad [\xi, \zeta^{(l,m)}] = \zeta^{(l,m-1)} . \tag{35}$$

The Cartan subalgebra of $\text{sl}(\infty; \mathbb{R})$ is spanned by the subset $\{\zeta^{(l,0)}, l = 1, 2, \ldots\}$ of symbols (34).

The above representation of $\text{sl}(\infty; \mathbb{R})$ in terms of symbols (34) is analogous to the construction of $\text{sl}(\infty; \mathbb{R})$ as "wedge" subalgebra $W_\wedge(\mu)$ of W_∞ for $\mu = 0$ [2,24], which in turn is isomorphic to the algebra A_∞ of Kac [25].

Now, accounting for (33), (34), one can write down explicitly the solution to the equations of motion (29):

$$Y_t(g^{-1})|_{\text{on-shell}} = \sum_{l=1}^{\infty} \sum_{|m| \leqslant l} J^{(l,m)}(t) \zeta^{(l,m)}(\xi, x) , \tag{36}$$

with $\zeta^{(l,m)}$ as in (34). The coefficients $J^{(l,m)}(t)$ in (36) are arbitrary functions of t and represent the on-shell form of the currents of the "hidden" $\mathscr{G}_{\text{stat}} \equiv \text{sl}(\infty; \mathbb{R})$ Kac–Moody symmetry of $W[g]$ (18).

[#2] The restriction of eq. (29) to the Virasoro subgroup via (14), (15) takes the well known form [16] $\partial_x^3(\partial_t f / \partial_x f) = 0$, and $f(x; t)$ is the inverse Virasoro group element: $f(F(x; t); t) = x$.

Indeed, upon right group translation with $\zeta_{(a)}=\sum a^{(l,m)}(x,t)\circ\zeta^{(l,m)}(\xi,x)$ with arbitrary coefficient functions (zero order symbols) $a^{(l,m)}(x,t)$, one obtains from (26)

$$\delta^{R}_{\zeta(a)}W[g]=-\frac{c}{4\pi}\int dt\,dx\sum_{l=1}^{\infty}\sum_{|m|\leqslant l}\partial_x a^{(l,m)}(x,t)\tilde{J}^{(l,m)}(x,t)\ , \tag{37}$$

$$\tilde{J}^{(l,m)}\equiv\sum_{r=1}^{\infty}(-\partial_x)^{r-1}\left((-1)^l l!(l+1)!\frac{x^m}{\Gamma(m)}[Y_t(g^{-1})]_r-\frac{1}{r+1}\partial_x[\zeta^{(l,m)}\circ Y_t(g^{-1})]_r\right). \tag{38}$$

The subscripts r in (38) and below indicate taking the coefficient in front of ξ^r in the corresponding symbol.

The Noether theorem implies from (37) that $\tilde{J}^{(l,m)}(x,t)$ (38) are the relevant Noether currents corresponding to the symmetry of the W_∞ action (18) under arbitrary right group $SL(\infty;\mathbb{R})$ translations. Clearly, $\tilde{J}^{(l,m)}(x,t)$ are $sl(\infty;\mathbb{R})$-valued and are conserved with respect to the "time-evolution" parameter $x\equiv x^-$:

$$\partial_x\tilde{J}^{(l,m)}(x,t)\,|_{\text{on-shell}}=0\ . \tag{39}$$

Substituting the on-shell expression (36) into (38) we get

$$\tilde{J}^{(l,m)}(x,t)\,|_{\text{on-shell}}=\sum_{l=1}^{\infty}\sum_{|m|\leqslant l}K^{(l,m)(l',m')}J^{(l',m')}(t)\ , \tag{40}$$

where $K^{(l,m)(l',m')}$ is a constant invariant symmetric $sl(\infty;\mathbb{R})$ tensor:

$$K^{(l,m)(l',m')}=\sum_{r=1}^{\infty}(-\partial_x)^{r-1}\left((\zeta^{(l',m')})_r\,\mathrm{Res}_\xi[\ln\xi,\zeta^{(l,m)}]-\frac{1}{r+1}\partial_x[\zeta^{(l,m)}\circ\zeta^{(l',m')}]_r\right), \tag{41}$$

naturally representing the Killing metric of $sl(\infty;\mathbb{R})$.

The fact that the currents $J^{(l,m)}(t)$ in (36) generate a $sl(\infty;\mathbb{R})$ Kac–Moody algebra, can be shown most easily by considering infinitesimal right group translation $g\to g\circ(\mathbb{1}+\zeta_\epsilon)$ with $\zeta_\epsilon=\sum_{l,m}\epsilon^{(l,m)}(t)\zeta^{(l,m)}(\xi,x)\in sl(\infty;\mathbb{R})$ on $Y_t(g^{-1})\equiv -g^{-1}\circ\partial_t g$. Recall (cf. (37), (40)), that $J^{(l,m)}(t)$ are the corresponding Noether symmetry currents. From the cocycle property (12) one obtains

$$\delta^{R}_{\zeta_\epsilon}Y_t(g^{-1})\equiv Y_t((\mathbb{1}-\zeta_\epsilon)\circ g^{-1})-Y_t(g^{-1})=-\partial_t\zeta_\epsilon+[Y_t(g^{-1}),\zeta_\epsilon]\ , \tag{42}$$

which upon substitution of (36) yields

$$\delta_\epsilon J^{(l,m)}(t)=-\partial_t\epsilon^{(l,m)}(t)+f^{(l,m)(l',m')}_{(l'',m'')}J^{(l',m')}(t)\epsilon^{(l'',m'')}(t)\ . \tag{43}$$

Here $f^{(l,m)(l',m')}_{(l'',m'')}$ denote the structure constants of $sl(\infty;\mathbb{R})$ in the basis $\zeta^{(l,m)}$ (34) (i.e., $[\zeta^{(l,m)},\zeta^{(l',m')}]=f^{(l,m)(l',m')}_{(l'',m'')}\zeta^{(l'',m'')}$).

Finally, let us also show that the canonical Noether energy–momentum tensor T_{++} (the Noether current corresponding to the symmetry of the W_∞ action (18) under arbitrary rescaling of $t\equiv x^+$) automatically has the (classical) Sugawara form in terms of the "hidden" $sl(\infty;\mathbb{R})$ Kac–Moody currents $J^{(l,m)}(t)$ (36). Indeed, the variation of (18) under a reparametrization $t\to t+\rho(t,x)$ reads

$$\delta_\rho W[g]=-\frac{1}{4\pi}\int dt\,dx\,\partial_x\rho(t,x)T_{++}(t,x)\ , \tag{44}$$

$$T_{++}\equiv\frac{1}{2c}\sum_{r=0}^{\infty}\frac{(-1)^r}{r+1}\partial_x^r\left([Y_t(g^{-1})\circ Y_t(g^{-1})]_r+\frac{1}{r!}\mathrm{Res}_\xi(\partial_\xi^{r+1}Y_t(g^{-1})\circ[\ln\xi,Y_t(g^{-1})])\right). \tag{45}$$

Substituting (36) into (45) and accounting for (34), one easily gets the $sl(\infty;\mathbb{R})$ Sugawara representation of the energy–momentum tensor (45):

$$T_{++}(t,x)\,|_{\text{on-shell}}=\frac{1}{2c}\sum_{(l,m),(l',m')}K^{(l,m)(l',m')}J^{(l,m)}(t)J^{(l',m')}(t)\,,\tag{46}$$

where $K^{(l,m)(l',m')}$ is the sl$(\infty;\mathbb{R})$ Killing metric tensor (41).

In particular, substituting into (45) the restriction of $g(\xi, x; t)$ to the Virasoro subgroup via (14), (15), we recover the well-known (classical) sl$(2;\mathbb{R})$ Sugawara form of T_{++} in $D=2$ induced gravity [26].

5. Conclusions and outlook

According to the general discussion in ref. [15], the Legendre transform $\Gamma[y]=-W[g^{-1}]$ of the induced W_∞-gravity WZNW action (18) is the generating functional, when considered as a functional of $y\equiv Y_t(g^{-1})$, of the quantum correlation functions of generalized schwarzians $S[g]$. Similarly, $W[J]\equiv-W_{\mathrm{DOP(S^1)}}[g]$, when considered as a functional of $J\equiv-(c/4\pi)S[g]$, is the generating functional of all correlation functions of the currents $Y_t(g^{-1})$. These correlation functions can be straightforwardly obtained, recursively in N, from the functional differential equations (i.e., Ward identities):

$$\partial_t\frac{\delta\Gamma}{\delta y}+\left[\frac{\delta\Gamma}{\delta y}-\frac{c}{4\pi}\ln\xi,y\right]_*=0\,,\quad \partial_t J+\left[\frac{\delta W}{\delta J},J-\frac{c}{4\pi}\ln\xi\right]_*=0\,.\tag{47}$$

An interesting problem is to derive the W_∞ analogue of the Knizhnik–Zamolodchikov equations [27] for the correlation functions $\langle g(\xi_1, x_1; t_1)\ \dots\ g(\xi_N, x_N; t_N)\rangle$. To this end we need the explicit form of the symbol $r((\xi,x);(\xi' x'))\in\mathcal{DOP}(\mathrm{S}^1)\otimes\mathcal{DOP}(\mathrm{S}^1)$ of the classical r-matrix of W_∞. This issue will be dealt with in a forthcoming paper.

Another basic mathematical problem is the study of the complete classification of the coadjoint orbits of DOP(S^1) and the classification of its highest weight irreducible representations.

Let us note that, in order to obtain the WZNW action of induced $\mathrm{W}_{1+\infty}$ gravity along the lines of the present approach, one should start with the algebra of differential operator symbols containing a nontrivial zero order term in the ξ-expansion $X=X_0(x)+\sum_{k\geqslant1}\xi^kX_k(x)$. In this case one can solve the "hidden" symmetry (i.e. the "anomaly" free subalgebra) equation $[\ln\xi,\zeta]_{(-)}=0$ and the result is the Borel subalgebra of gl$(\infty;\mathbb{R})$ spanned by the symbols $\zeta^{(p,q)}=\xi^px^q$ with $p\geqslant q$. The $\mathrm{W}_{1+\infty}$ WZNW action will have formally the same form as (18), however, now the meaning of the symbol $g^{-1}(\xi, x; t)$ of the inverse group element is obscure due to the nontrivial ((ξ,x)-dependent) zero order term in the ξ-expansion of $g(\xi, x; t)$.

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